The smarter, the cheaper[1]

Aging, bounded rationality and the natural rate of interest

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Abstract

This paper revises the secular stagnation hypothesis through the lens of bounded rationality. The consequences of population aging on medium and long-term equilibrium are at the center of today's discussion of macroeconomics. According to the secular stagnation hypothesis in aging societies the GDP growth decelerates and the natural rate of interest decreases when the households cumulate more savings for the longer lifetime span. However, this model prediction can be rejected or weakly explained by historical data of OECD countries. To the best of my knowledge, this is the first paper on multi-period Gertler-type OLG-model that incorporates bounded rationality and empirically shows that negative relationship between old-age dependency ratio and real interest rate can only be valid for those countries where the agents' behavior is consistent with the rational expectation equilibrium or the agents have relatively long planning horizon.

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"By the time you're eighty years old you've learned everything. You only have to remember it."

George Burns (1896-1996)

1 Introduction

In the last decades the aging problem has prevailed in the developed and emerging economies, more and more empirical and theoretical papers discussed the economic effect of demographic changes. According to the *secular stagnation* literature the slower economic growth is coupled with the fall of natural rate of interest (Ferrero et al, 2017)[3]. The latter phenomenon is a strong and robust prediction of the standard DSGE-OLG models either, however the empirical findings for the long-term interest are not necessarily consistent with the theoretical papers. In a global VAR model it can be shown that the demographic trends have a negative impact on the long term interest rate, however, these results are not robust for two-ways estimation (Aksoy et al, 2019). If one controls on the time fixed effect (common latent component of long term interest rate), the effect of demographic variables becomes insignificant. Others have same conclusion for the European population aging, and claim that the demographic trends in Europe do not support the secular stagnation hypothesis and the expected age-structure of the population will generate positive real rates in the future (Favero and Galasso, 2015). In this paper I estimate a panel model on OECD countries between 1992 and 2017, and I can also show the effect of demographic variables become insignificant once I use a two-ways estimation method (involving country and time fixed effects at the same time). This result challenges the prediction of the standard DSGE-OLG framework, however my findings are consistent with the recent empirical papers above.

In this paper based on the simplified version of OGRE model (Overlapping Generations for Retirement, Baksa-Munkácsi (2016 and 2019)) I compare the traditional and bounded rational prediction of OLG-models for the natural rate of interest at the time of population aging. The current version assumes a simple, frictionless one sector economy. The households' behavior can be described by the Gertler-type OLG framework. The young generation supplies labor, pays lump-sum taxes to the government and owns the firm of the economy. The old, retired households receive pensions from the pay-as-you-go (PAYG)

[3] According to the New-Keynesian terminology the interest rate of the flexible price equilibrium is called natural rate of interest.

pension system. Both cohorts are able to save or take credit and finance the public debt. The aging shock increases the lifetime span of retired cohort and decreases fertility rate by young households. According to the standard rational expectation theory, at the time of aging the agents change their saving or credit position. The households by the increasing lifetime span decide to cumulate more savings and therefore adjust their consumption. The young households also anticipate the increasing future financing need of the public pension system and the increasing level of private savings exert negative pressure on the natural rate of interest. The bounded rationality changes the young households' savings attitude and makes them relatively more indebted that results higher long run interest rate than in full rational equilibrium.

According to the bounded rationality the agents have cognitive limits, their future expectation is distorted, and then the households' consumption and savings decision significantly differs from the rational expectation equilibrium. In behavioral macroeconomics the level-k thinking is the common way to model bounded rationality[4]. One can show that the level-k thinking is the special case of myopia, and the size of k can be interpreted as the length of the planning horizon[5] (Lovo, 2000). In these settings the households are able to take into account only the first k periods of future information, after period k the households expect that the economy will revert back to the initial steady-state or to the initial balanced growth path equilibrium. The biased expectation channel could be crucial if the economy is affected by permanent demographic shocks. Despite of that the population aging generates continuously increasing financing problem in the pension system, the agents with bounded rationality are less careful about their own future wealth and more serious fiscal issues. With relatively low k the young households consider the increasing taxes or debt financing as a temporary economic event, hence they do not adjust their permanent income and consumption. In addition, they take more credit to avoid welfare loss, but the higher credit demand elevates the natural rate of interest that increases the financing costs and incites the retired cohort to cumulate even more savings. In the paper I compare the demographic transition with different k-s and also compare to the results of rational expectation equilibrium.

[4]There are other types of non-rational models. According to Sims (2003) there are three categories for non-rational models: (1) behavioral economics literature; (2) learning literature; (3) robust control literature and Sims (2003) suggested another direction either, where the people have limited capacity for processing information.

[5]In this framework the rational expectation can be interpreted as a special case of bounded rationality when the agents consider all available information about the future.

The paper also tests the empirical relationship between the demographic component and long-term real interest rate for the OECD countries. In the first *naive* estimation, I show that the negative relationship is not necessarily robust for all countries. If one uses two-ways estimation, and involves common time fixed effect, the secular stagnation hypothesis does not hold for all OECD countries and the estimated parameters become insignificant. This result is consistent with Aksoy et al (2019). Nevertheless, the bounded rationality sheds light to the weakness of naive empirical investigations. By the initial two-ways estimation there is no control on any selection bias, and implicitly it is assumed that all OECD countries follow the same (fully rational) consumption and savings behavior. However, the model with bounded rationality implies that the negative relationship between demographic factors and interest rate is only true for those economies where the agents' expectation is close to the rational expectation. Therefore, in the further exercises I adjust the *naive* estimation involving interaction terms to select the countries according to being rational and non-rational. For the rationality I used two different proxies: (1) the financial literacy indicator (Klapper et al, 2015); and (2) time preference from Global Preference Survey (Falk et al, 2018). In both specification the estimated coefficients of the demographic factor are significant and have reasonable, negative value only for the countries considered financially literate or have higher index for time preferences. The estimated coefficients are robust to the time-range and also for dynamic panel setup. These results confirm the prediction of bounded rational OLG-models, namely the natural rate of interest is expected to be decreasing only in those countries where the agents' expectation is close to the rational case.

In the rest of the paper section 2 gives a review of related literature and defines my contribution, section 3 describes the benchmark-model, discuss the phenomenon of secular stagnation and shows the results of naive estimations. Section 4 compares the rational and bounded rational equilibrium outcomes and long-term properties. Finally, in section 5 controlling on rational behavior via the interaction terms I re-estimate the panel model and check the robustness of the results.

2 Literature review

Since the great recession, and due to the elongated economic recovery, the secular stagnation has become one of the most relevant topic in todays macroeconomics. The post-crisis US recovery with decelerated productivity growth and population aging results slower potential growth and historically low interest rate, however according to Summers (2014) the effect of demography is negligible. In the last decades many further papers rediscovered the secular stagnation, revised the effect of population aging on gloomy recovery and period of low interest rate.

In the literature the overlapping-generation (OLG) models are the common tool to understand and predict the future effect of population aging, however the most recent empirical studies are not fully consistent with the theory-based models and lead to different conclusions. The OLG-models have the same long history as the real-business cycle models, Auerbach and Kotlikoff (1987) in a consumption life-cycle model examined the medium-term effect of different tax policies and demographic changes, their paper later served as a reference point for further researches. Their model belongs to the Diamond-style OLG framework, where the households are distributed in well defined cohorts and are assumed to have fixed life-time horizon. In another type of OLG models instead of explicit age cohorts it is assumed that the average life-time of households can be expressed by survival rates (Blanchard, 1985; Yaari, 1965). These models also dissolves the Ricardian equivalence[6], and through the intergenerational redistribution, the fiscal policy can directly influence the agents' behavior. It is also possible to adjust the Blanchard-Yaari specification with an additional cohort to separate the worker and retired households[7] according to their labor market participation and eligibility for pension benefit (Gertler, 1999). The Blanchard-Yaari-type and Gertler-type models have very similar structure like the dynamic stochastic general equilibrium (DSGE) models, and they can be easily combined with the New-Keynesian features. However, until the great recession these models were not at the center of macroeconomic discussions. The slow recovery, gloomy demographic outlook, and later the emerging fiscal imbalances favored the alternative, non-Ricardian interpretation of fiscal policy. The Global Integrated Monetary and Fiscal Model (GIMF) among the first implemented Blanchard-Yaari version of OLG models (Kumhof et al, 2010), and later in many other papers included these types of non-Ricardian features. The fundamental differences between the two models is the description of households behavior. While in the DSGE-models the representative households' consumption function implicitly can be described by the Euler-equation, in the OLG-models for each individuals, one should ex-

[6]Ricardian equivalence is a common property of DSGE or RBC models, that is caused by the forward-looking rational behavior, infinite planning horizon and lack of liquidity constrains. According to the equivalence theorem the households are neutral regarding the fiscal redistribution and there is no difference between the timing of tax increase or domestically financed debt issuance.

[7]The workers with given probability become retired, and the retired households survive each period with given probability.

plicitly solve individual consumption function and these functions can be aggregated to express cohort level consumption.

One of the main focus of the current OLG literature is the current and expected position of the natural rate of interest. The demographic transition has a significant impact on real economic variable and the long-term interest rate (Carvalho et al, 2016). The demographic trend explains 1.5 percentage point of the interest rate decline between 1980 and 2030, and one has also shown that the decrease of natural rate of interest may contribute to the deflationary pressure, if the central bank is not able to follow the flexible price consistent interest rate (Bielecki et al, 2017). Others also verify that in the US the demographic factors contributed to the sluggish recovery and depressed monetary conditions (Gagnon et al, 2016; Eggerston et al, 2017; Jones, 2018).

Despite of the similar theoretical findings the empirical results are less unambiguous. Ferrero et al (2017) found relationship between demographic factors and interest rate. According to Arslanalp et al (2017) in Asian economies the interest rate and demography is well connected. However, in Europe the real interest rate will recover, and because the secular stagnation hypothesis is not valid the long-term interest will not be decreasing (Favero and Galasso, 2015). Aksoy et al (2019) showed in a panel VAR model that the demographic structure has effect on the medium-term growth and long-term yields, although their results for natural interest rate is not robust for the two-ways estimation. I will demonstrate later, in a simple *naive* panel estimation that the relationship between demographic factor and long term real interest rate disappears, as one involves the time-fixed effect into the estimation. In the rest of the paper I show possible explanations for the misspecification and based on the bounded rationality extension I give more insight how to adjust the empirical estimation.

As the great recession elongated, the new flow of economic theories, as well as the behavioral macroeconomics, have become more popular. Conlisk (1996) summarized the main advantage and reasons of models with bounded rationality. The concept of level-k thinking is used from Fair and Taylor (1983) and Evans and Ramey (1992; 1995; 1998), these papers have shown how different equilibrium can be calculated from iteration process, and compared the rational and bounded rational behaviors. It can be shown that the bounded rationality is equivalent with the myopic behavior and can be linked to the length of the planning horizon (Lovo, 2000). Despite of non-rational behavior in these models the Ricardian equivalence proposition can continue to hold, the existence of equivalence depends on the government transversality conditions (Evans et al, 2010). Fahri and Werning (2017) derived analytically how the interest rate elasticity can change if the agents have bounded rationality with level-k thinking and occasionally-binding borrowing constraints. Gabaix (2018) described the properties of behavioral New-Keynesian model and compared the impulse responses of typical macroeconomic shocks, however his paper did not focus on fiscal policies. Park and Feigenbaumz (2018) shows that the life-cycle model with bounded rationality can generate hump-shaped consumption profile that matches to the US data.

My paper has several contributions to the macroeconomic literature. To the best of my knowledge, this is the first paper that merges the Gertler-type OLG models with bounded rationality and level-k thinking to examine the population aging with non-rational expectations. My paper complements to Fahri and Werning (2017) and Gabaix (2018) with the focus on fiscal policy, demographic shock and overlapping generation framework. It exceeds Parkyand and Feigenbaumz (2018) either because my paper describes the general equilibrium effect on natural interest rate and I do not assume time-invariant long-term interest rate. Finally, this extension gives economic evidence and more insight for the empirical identification strategy, if one wants to estimate the relationship between the demography and long-term interest rate. I show that the negative coefficient between demographic factors and interest rate is robust and significant for those economies only where the households' behavior is consistent with the rational expectation equilibrium.

3 Secular stagnation hypothesis and Overlapping Generations

This section describes a benchmark Gertler-type overlapping generation model and the secular stagnation hypothesis. The model is the simplified version of Baksa-Munkácsi (2016 and 2019).

3.1 An OLG-model a la Gertler

In the Gertler-type OLG models there are two cohorts (workers and retired). The workers arrive to the cohort with the rate n, and become retired in the next period with probability ω^Y. The retirees pass away with probability ω^O. The households in each cohort are able to save or consume from their disposable income. The workers earn wage income (wL) from the firms of the economy, and as the owner of the firms they also receive the profits. The government is responsible for the pay-as-you-go (PAYG) pension system, where the benefit is the function of the wage income from the pre-retirement period and the exogenous replacement rate (ν). At the time of retirement based on previous wage income flow the government calculates the just-retired pension and supplies this benefit until the death of the retiree. The government also issues risk-free bond that is cumulated by both households. In the overlapping generation models the interest rate is the explicit function of bonds market equilibrium, that means the steady-state interest rate is not given by the inverse of time-preference from the households' utility function. Below I summarize the behavioral equation of the model.

Old-age dependency ratio s, the number of the retired population divided by working age population, can be given by the survival rates, fertility rates and the previous value of old-age-dependency ratio. s^Y, the relative size of worker cohort, can be given by the function of old-age dependency ratio:

$$\begin{aligned} s_t &= \frac{(1-\omega_{t-1}^O)}{(1-\omega_{t-1}^Y+n_t)} s_{t-1} + \frac{\omega_{t-1}^Y}{(1-\omega_{t-1}^Y+n_t)} \\ s_t^Y &= \frac{1}{1+s_t} \end{aligned}$$

The cohort level ($g^{N,Y}$ is the growth rate of worker cohort, $g^{N,O}$ is the growth rate of retired cohort) and total population growth (g^N) can be defined as the function of survival rates and old-age dependency ratio:

$$\begin{aligned} 1+g_t^{N,Y} &= 1-\omega_{t-1}^Y+n_t \\ 1+g_t^{N,O} &= (1-\omega_{t-1}^O) + \frac{\omega_{t-1}^Y}{s_{t-1}} \\ 1+g_t^N &= (1+g_t^{N,Y}) \frac{1+s_t}{1+s_{t-1}} \end{aligned}$$

The dynamic optimization of individuals can be described by Bellman-equation, where the maximizing utility function is the combination of the individual consumption and leisure (the leisure matters for the workers only). Due to the overlapping generation framework, the first order conditions do not describe the representative behavior of the households, because the agents are born in different periods and can have different wealth position. Therefore, beyond the first order condition, we need to express the individuals' explicit consumption function from the Euler equation and intertemporal budget constraint, and in the final step of the derivations with the sum of all individuals' consumption and bond one can express the cohort level aggregate consumption and law of motion for bonds. In growth models it is common that the non-price related variables are expressed in terms of balanced growth trend[8]. Then the aggregate per capita variables are function of the demography. The aging shock directly affects the balanced growth path and the short run dynamics of the normalized variables.

Following the logic above, first one can derive the consumption function of retirees and later the workers' function. The retirees' consumption (C^O) is the function of the expected permanent income, the initial bonds of the survived retired population (B^O), the inherited bond of just-retired (B^Y) and the marginal propensity to consume (MPC^O). The permanent income is the function of the actual pension (TR), and the time-variant discount factor of retired cohort (Ω^O), that takes into account the current and expected real interest rate and the probability of death. The retired cohort consumption function is the following:

$$\tilde{C_t^O} = MPC_t^O \tilde{TR}_t \Omega_t^O + MPC_t^O \frac{(1+r_{t-1})}{1+g_t^N}\left(\omega_{t-1}^Y \tilde{B_{t-1}^Y} + \tilde{B_{t-1}^O}\right)$$

where

$$\Omega_t^O = 1 + E_t \frac{1-\omega_t^O}{1+r_t}\Omega_{t+1}^O$$

$$\frac{1}{MPC_t^O} = 1 + E_t(1-\omega_t^O)(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\frac{1}{MPC_{t+1}^O}$$

γ is the inverse of intertemporal elasticity from the households' utility function, β is the time-preference, r is the real interest rate.

Pay-as-you-go (PAYG) pension system and pension expenditures are function of two components: (1) the just-retired initial pension that is linked to the pre-retirement labor income (wL) and the exogenous replacement rate (ν); (2) old-retireds' pension:

$$\tilde{TR}_t = \nu \frac{\omega_{t-1}^Y}{1+g_t^N} w_{t-1}\tilde{L}_{t-1} + \frac{(1-\omega_{t-1}^O)}{1+g_t^N}\tilde{TR}_{t-1}$$

The pension expenditures are connected to biological factors via the survival rate, then the increasing longevity (decreasing ω^O) ceteris paribus generates higher fiscal expendi-

[8] The $\tilde{x}_t$ denotes the value of x_t normalized by the balanced growth path at period t. In this version of the model the balanced-growth path is function of the population growth, and I do not assume productivity growth.

tures. Because any intervention to pension expenditures can go through the just-retired benefit, the pension reforms can slowly stabilize the fiscal balance. The workers' behavior can be also described by a dynamic optimization problem, however their utility function contains leisure, and σ shows the relative importance of consumption in utility function. The individual decision takes into account the probability of the next period retirement (ω^Y) and the expected value of the future labor income. Assuming state-contigent bonds the households do not know the exact time of their own retirement, but at the time of retirement all of their previously cumulated wealth is transferred into their retired-self balance sheet and then the individuals are able to smooth out their consumption. After the aggregation and normalization, the workers' cohort-level consumption function (C^Y) can be written as

$$\tilde{C_t^Y} = MPC_t^Y \tilde{Inc_t} + MPC_t^Y \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t^N} \tilde{B_{t-1}^Y}$$

where the expected income is the function of the current disposable income, and the ω^Y weighted next period retired income or $1-\omega^Y$ weighted future net labor income:

$$\tilde{Inc_t} = w_t s_t^Y + \tilde{Profit_t} - \tilde{Tax_t} + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc_{t+1}}$$

The labor supply curve can be derived from the first-order condition:

$$\frac{\tilde{C_t^Y}}{s_t^Y - \tilde{L}_t} = \frac{\sigma}{1-\sigma} w_t$$

The marginal propensity to consume (MPC^Y) is the function of the real interest rate, the weighted average next period MPC-s:

$$\frac{1}{MPC_t^Y} = \frac{1}{\sigma} + E_t(1+r_t)^{\frac{1}{\gamma}-1} \left[(1-\omega_t^Y)\Lambda_t^Y \frac{1}{MPC_{t+1}^Y} + \omega_t^Y \Lambda_t^{YO} \frac{1}{MPC_{t+1}^O} \right]$$

For simplification we assigned additional two variables from the equation of MPC^Y:

$$\Lambda_t^Y = \beta^{\frac{1}{\gamma}} \left(E_t \frac{w_{t+1}}{w_t} \right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)}$$

$$\Lambda_t^{YO} = \left\{ \frac{\beta}{\sigma} \right\}^{\frac{1}{\gamma}} \left(\frac{1}{\frac{\sigma}{1-\sigma} w_t} \right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)}$$

Due to the two distinct cohorts and two bonds, from the workers' period-t budget constraint one can express the law of motion for risk-free bonds:

$$\tilde{B_t^Y} = w_t\tilde{L}_t + Pr\tilde{o}fit_t - T\tilde{a}x_t - \tilde{C_t^Y} + \frac{(1+r_{t-1})}{1+g_t^N}(1-\omega_{t-1}^Y)\tilde{B_{t-1}^Y}$$

The profit-maximizing firms have the usual Cobb-Douglas production function and for simplicity I assume that these firms are price-takers, then their profits are zero. Production function can be given by

$$\tilde{Y}_t = A_t\left(\frac{\tilde{K}_{t-1}}{1+g_t^N}\right)^{\alpha}\tilde{L}_t^{1-\alpha}$$

The firms are also responsible for capital accumulation, the law of motion for capital is the following

$$\tilde{K}_t = I\tilde{n}v_t + (1-\delta)\frac{\tilde{K}_{t-1}}{1+g_t^N}$$

where δ is the depreciation of capital, Inv is the level of private investment. Labor demand and implicit capital demand functions can be given by the following equations:

$$w_t = (1-\alpha)\frac{\tilde{Y}_t}{\tilde{L}_t}$$

$$1+r_t = E_t\alpha(1+g_{t+1}^N)\frac{\tilde{Y}_{t+1}}{\tilde{K}_t} + (1-\delta)$$

The profit can be given as

$$Pr\tilde{o}fit_t = \tilde{Y}_t - w_t\tilde{L}_t - I\tilde{n}v_t$$

Following the New-Keynesian terminology the flexible price assumption implies that the total output and the real interest rate can be interpreted as the potential output and the natural rate of interest.

The government budget constraint describes the law of motion for public debt, that is ceteris paribus increasing if the government expenditures exceed the tax revenues, or the government has to pay higher interest-cost:

$$D\tilde{e}bt_t = T\tilde{R}_t + G\tilde{o}v_t - T\tilde{a}x_t + \frac{(1+r_{t-1})}{1+g_t}D\tilde{e}bt_{t-1}$$

where the government consumption (Gov), taxes (Tax) or the debt level ($Debt$) can be exogenously given. It is also possible to implement a fiscal rules that anchors the variables (see Baksa-Munkácsi et al (2016 and 2019)).

In the OLG models the bond market equilibrium should be satisfied. The bonds market equilibrium is an essential part of the equilibrium conditions because it determines the equilibrium interest rate:

$$\tilde{Debt}_t = \tilde{B}_t^Y + \tilde{B}_t^O$$

The usual goods market equilibrium is the following:

$$\tilde{Y}_t = \tilde{C}_t^Y + \tilde{C}_t^O + \tilde{Inv}_t + \tilde{Gov}_t$$

The model can be simplified into two main equations that characterize the transitional dynamics and steady-state equilibrium. Based on the both cohorts' consumption function, labor supply curve and λ-s one can derive dynamic IS-curve that explicitly describes the workers' bonds supply curve:

$$\begin{aligned}\left(\frac{1}{MPC_t^Y} - \frac{1}{\sigma}\right)\tilde{C}_t^Y &= \underbrace{\tilde{B}_t^Y\left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}}\right)}_{\text{Wealth effect}} + \underbrace{E_t\frac{\omega_t^Y}{1+r_t}\nu\alpha\tilde{Y}_t\Omega_{t+1}^O}_{\text{Expected pension}} + \\ &\quad + \underbrace{E_t\frac{1-\omega_t^Y}{1+r_t}\frac{1+s_{t+1}}{1+s_t}\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}}_{\text{Workers' expectation}}\end{aligned}$$

where the 'Wealth effect' assigns the direct effect of the accumulated bonds on workers' consumption. Due to overlapping generations 'Expected pension' also influence the non-retired behavior, however its effect on expectation is relatively small. 'Workers' expectation' denotes the effect of workers' expectation on the young cohort's consumption. It can be shown if $\omega_t^Y = 0$ for all t and there is no retirees in the economy, this equation collapses into the standard Euler-equation of the representative real business cycle models.

The demand for the workers' bond can be expressed as the following:

$$
\begin{aligned}
\tilde{B}_t^Y \quad = \quad & \underbrace{\tilde{Debt_t}}_{\text{Public debt}} - \underbrace{(1 - MPC_t^O \Omega_t^O)\tilde{TR}_t}_{\text{Savings from pension}} - \\
& -\frac{(1+r_{t-1})}{1+g_t^N}(1 - MPC_t^O)\underbrace{\left[\tilde{Debt}_{t-1} - (1-\omega_{t-1}^Y)\tilde{B}_{t-1}^Y\right]}_{\text{Non-consumed pensioner savings}}
\end{aligned}
$$

where 'Public debt' is the actual level government debt, total demand for all households' savings; 'Savings from pension' is the non-consumed part of the life-time income of pensioners; 'Non-consumed pensioner savings' denotes the reinvested part of the period $t-1$ retirees' bonds. The latter two components denotes the current savings of retired cohort, the rest of the debt should be covered from the workers savings.

These two equations above determine the optimal consumption-savings position of the workers cohort. With the labor-supply curve, production function and good market equilibrium one can derive the rest of the other variables. Rest of the parameters and steady-state ratios are available in the appendix.

3.2 Population aging and secular stagnation

In this part I show the conventional prediction of OLG models about the population aging. First, the response function describe the medium term accommodation, in the next phase I compare the initial and terminal steady-state points.

At the time of population aging the longevity increases and the fertility (the arrival rate of new labor force) decreases (see figure 1). The aging distorts the demographic distribution, and the increasing old-age dependency ratio also indicates that the size of retired cohort exceeds the initial level. The increasing longevity in a PAYG pension system automatically means that the pension expenditures will increase and the government needs to issue more public debt or it will increase the taxes[9]. In this simulation, I assume that the fiscal policy raises the taxes to fully offset the debt increase[10]. The decline in the working age population results lower GDP per capita, and lower domestic demand. The workers realize the longer life-time span and are also aware of that in the future they need

[9] This paper does not examine the effect of other pension reforms eg.: increase of retirement age, increase of contribution rate or decrease of replacement rate. In Baksa-Munkácsi (2016) and forthcoming Baksa et. al. (2019) paper we exam the macroeconomic effect of various pension reforms in different European economies.

[10] In the closed economy model most of the government debt is financed by the workers cohort, then the increase of lump-sum taxes or the increasing public debt generates similar outcome.

to pay more taxes to the government, therefore from the first moment of the transition they adjust their consumption according to the updated permanent income expectation. The increasing amount of savings are transferred into capital investment, or if the fiscal policy does not change the taxes, the young households cumulate even more bonds to finance the increasing public debt. Due to the quick accommodation of the young households, the higher level of private savings exerts negative pressure on the real interest rate and then on marginal product of capital. The retired households on longer life-time horizon save more to secure their individual consumption. Although the retirees become more cautious on individual level, on the aggregate level the retired cohort receive relatively more benefits and the weight of the old-age cohort increases, then the aggregate level consumption of the retired cohort increases at the time of aging.

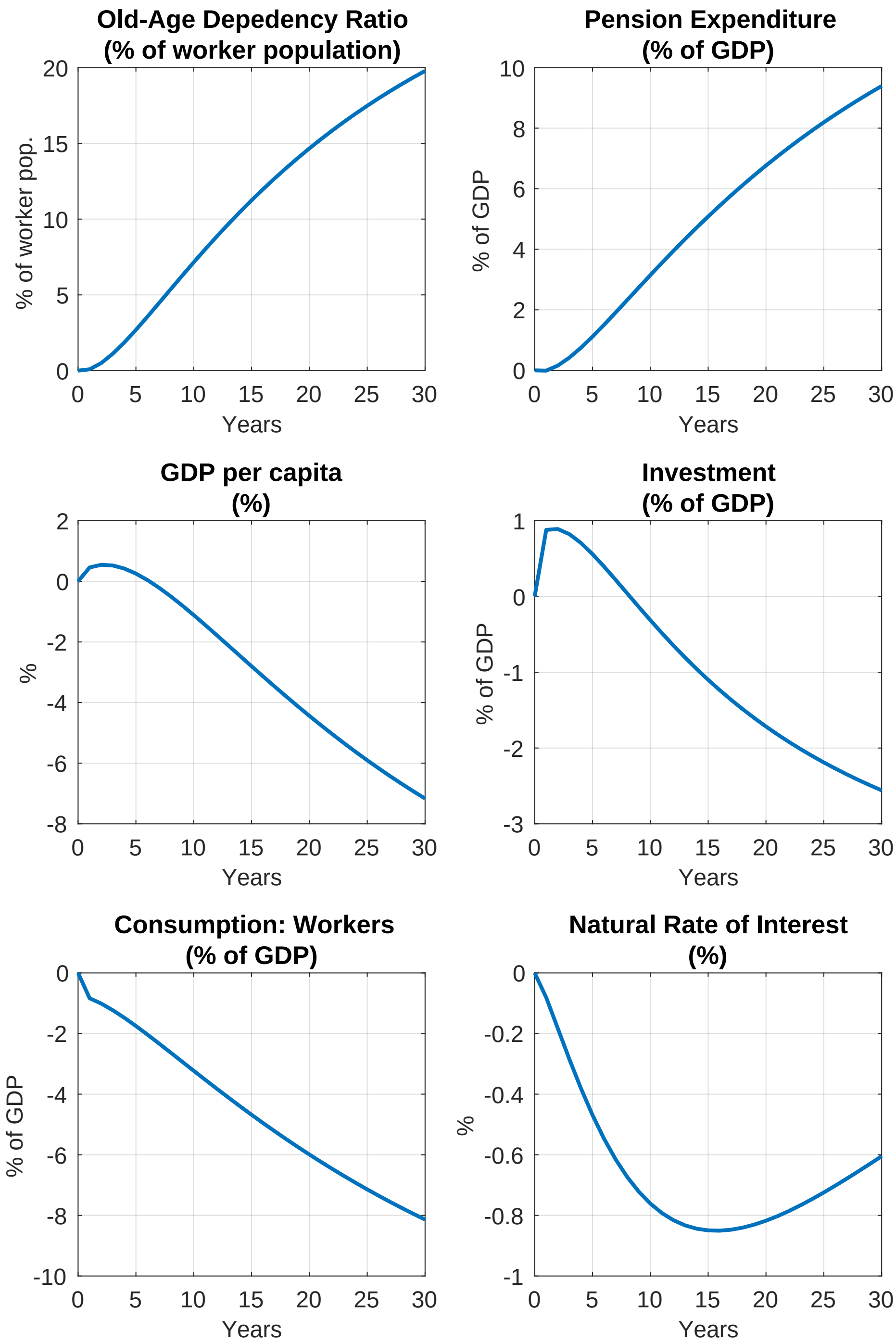

Figure 1: Population Aging and transitional dynamics in rational expectation equilibrium

The population aging has permanent effect and also changes the long-term, steady-state, position of the economy (see figure 2). Based on the dynamic IS-curve equation and demands for young bonds one can express the workers' consumption in GDP terms as the function of real-interest rate, the intersection of solid lines is the initial, the dashed lines give the terminal steady-state equilibrium. The aging shock ceteris paribus increases the pension expenditures. The expected tax increase or the higher demands for worker savings shifts the young bonds curves, which ceteris paribus increases the real interest rates. Nevertheless, the worker households decreases their consumption and saves more, this reaction shifts down the dynamic IS-curve to the new long-run equilibrium point. Because of the capital accumulation the IS-curve is more downward sloping. In the new long-run equilibrium the real interest rate is higher than initially. This terminal steady-state is consistent with the fact that in aging societies on the long run the capital stock is lower, and due to the scarcity of capital the marginal product of capital increases that has a positive effect on the long run real interest rate either.

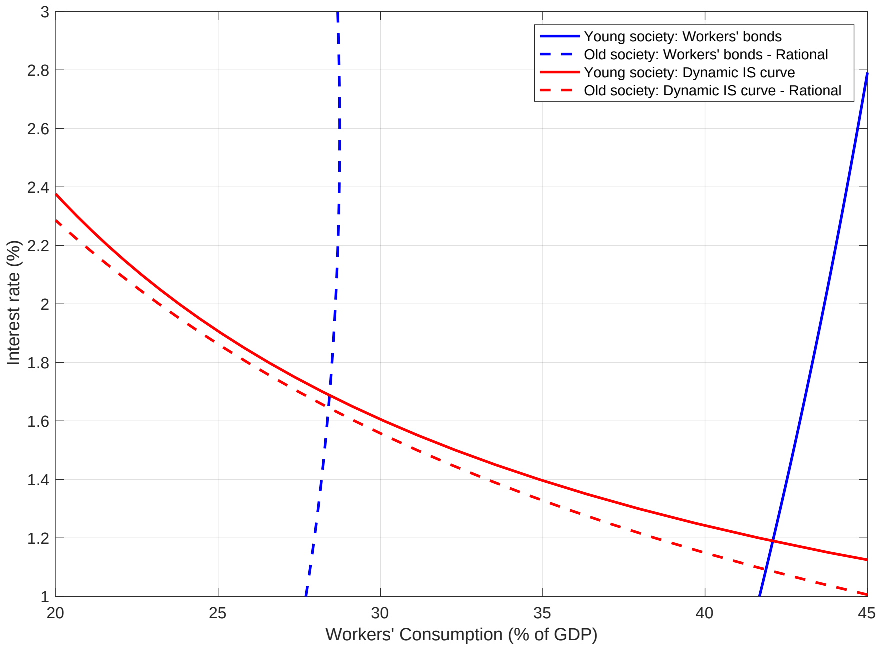

Figure 2: Initial and new steady-state equilibrium in rational expectation equilibrium

3.3 Naive estimation: testing the secular stagnation hypothesis

Previously I have shown the effect of population aging in a benchmark OLG-model. It is essential to check whether the last 20 years of macroeconomic data supports the idea of secular stagnation. Figure 3 describes the historical development of the average old-age dependency ratio and average real interest rate with 90% confidence band. It suggests a negative relationship between the demographic component and real interest rate.

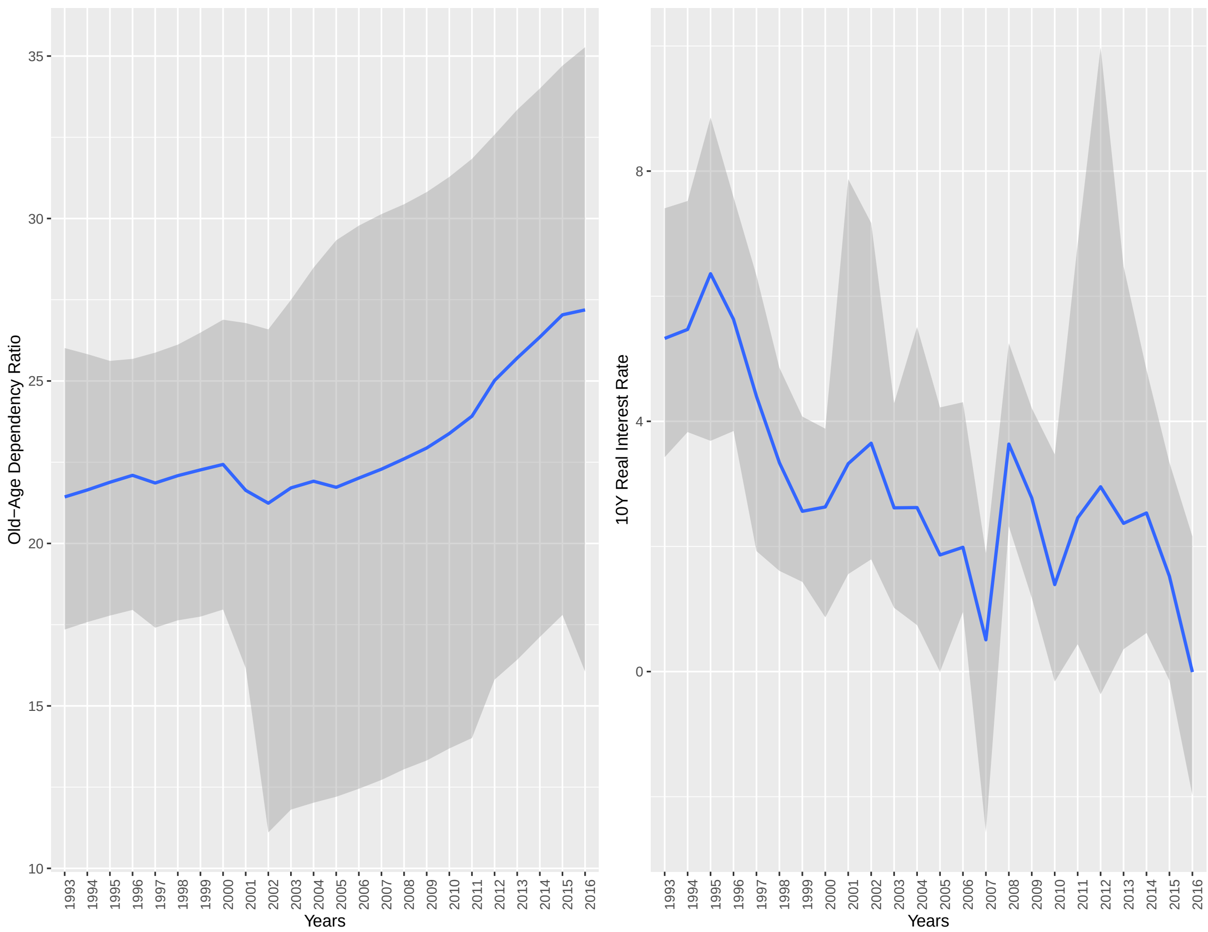

Figure 3: Old-Age Dependency Ratio and Real Interest Rate between 1993 and 2016

To check this co-movement, I estimate a simple unbalanced panel model on the OECD countries between 1992 and 2016[11]:

$$r_{it} = \rho \cdot r_{i,t-1} + \alpha_i + \delta_t + \gamma \cdot OADR_{it} + u_{it}$$

[11]The starting year was arbitrary, although most of the OECD countries published data from the beginning of 1990s, and the population aging become more evident in the last decade of the 20th century.

where r_{it} 10Y nominal yields minus inflation from OECD database, $OADR$ old-age dependency ratio (65+ over 20-64 years old population) from UN database, α_i country fixed effect, δ_t time fixed effect. The static and dynamic model were estimated in R with plm package (Croissant and Millo, 2008) [12], and I calculated robust standard errors.

	Dependent variable:					
	r_{it}					
	FE	FE	FE	FE	FE	FE
	(1)	(2)	(3)	(4)	(5)	(6)
r_{it-1}		0.45	0.42		0.49	0.49
		(0.09)***	(0.1)***		(0.09)***	(0.09)***
$OADR_{it}$	−0.35	−0.26	−0.08	−0.06	−0.03	−0.03
	(0.08)***	(0.06)***	(0.04)*	(0.09)	(0.05)	(0.06)
r_t^{US}			0.6			−0.14
			(0.1)***			(0.13)
Country FE	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	No	No	No	Yes	Yes	Yes
Observations	483	482	458	483	482	458
R^2	0.13	0.33	0.41	0.39	0.56	0.56
Note:						*p<0.1; **p<0.05; ***p<0.01
						Robust standard errors in parentheses

Table 1: Naive estimation: Long-term real interest rate and demography

The fixed-effect estimations are reported in table 1. In this exercise I estimated different specification to test the robustness of the negative relationship between demography and long term interest rate. The FE (1)-(2) specifications are the common naive estimation that results in a significant negative relationship between the old-age dependency ratio and the real interest rate. The static and dynamic panel shows the same sign for the coefficient. It is not possible to involve more lag for the old-age dependency ratio, because the demographic variables are really smoothed and strongly autocorrelated, therefore any additional lag in the estimation generates a strong multicollinearity in the estimation. According to the first two estimations, the demographic components are very good predictors of the long-term real interest rate and they are seemingly consistent with the OLG-models. However, these results are not robust for the two-ways estimation. The FE (5) static and FE (6) dynamic panel assumes time fixed effect component, then the explanatory power of demographic factors on real interest rate completely disappears. This result is consistent with Aksoy et al (2019). Based on these specifications the changes in demographic trend do not generate automatic decline in interest rate, and in many countries the fall of the long term interest

[12]The dynamic panel model was estimated with within estimator and also with GMM (Arellano and Bond (1991)), but I have not found significant differences between the two results

rate is not caused by the population aging. It suggests that in many countries the fall of real interest rate is sourced from the international spillover effects (that can be estimated by the common time component). To check how the common time component are linked to the global financial markets, I added the US interest rate separately into the estimation as a common observation and estimated the following equation (with and without time fixed effect):

$$r_{it} = \rho \cdot r_{i,t-1} + \alpha_i + \delta_t + \gamma \cdot OADR_{it} + \xi \cdot r_t^{US} + u_{it}$$

where the r^{US} is the US long term real interest rate for all countries. FE (3) and FE (6) report the results of the estimation, by this specification I dropped the US country-level observations from the sample. In FE (3) once I involve the US long term real interest rate instead of time fixed effect, the p-value of demographic components becomes weaker and the size of the coefficients decreases in absolute terms, the coefficient of US real interest rate is positive and significant. By FE (6) I tested whether the US real interest rate has an additional explanatory power with the time fixed effect. In this case the US rates lost their explanatory power. These findings suggest that the US real interest rate could be a potential common components, and contrary to the benchmark OLG-model the negative relationship between demographic factors and long-term interest rate is not necessarily true for all OECD countries. Nevertheless, the US will not have such a serious aging problem like the European countries, then the decline in the US rates is sourced from the effect the *Great Moderation*[13] and the period of post-crisis unconventional monetary policy.

4 Bounded rationality and OLG-models

Conlisk (1996) gives a nice summary and insight about the advantages and empirical relevance of bounded rationality. He summarizes that many evidence support the idea of that the optimizing agents have cognitive limits and are not able to process all information by their economic decision. In other words the agents' expectation are distorted under the bounded rationality and this bias generates different outcome compared to the rational expectation equilibrium.

In the simplified Gertler-type OLG-model the two main equations anchor the equilibrium, where the expectation terms are distorted in some way. Later I give a formal definition of the distorted expectation, and describe the level-k thinking. As a first step I

[13]The period of low inflation and low interest rate is called 'Great Moderation'.

want to show that how any distortion or error in the expectation can deviate the model behavior from the rational expectation equilibrium.

Formally it means that the dynamic IS-curve can be written as

$$\left(\frac{1}{MPC_t^Y} - \frac{1}{\sigma}\right)\tilde{C}_t^Y = \tilde{B}_t^Y\left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}}\right) + \widehat{E}_t\frac{\omega_t^Y}{1+r_t}\nu\alpha\tilde{Y}_t\Omega_{t+1}^O + $$
$$+\widehat{E}_t\frac{1-\omega_t^Y}{1+r_t}\frac{1+s_{t+1}}{1+s_t}\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}$$

where $\widehat{E}_t\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}$ denotes the biased expectation. The old-age discount factor influences the retirees' consumption and the demand for workers' bond the old-age discount factor

$$\Omega_t^O = 1 + \widehat{E}_t\frac{1-\omega_t^O}{1+r_t}\Omega_{t+1}^O$$

where $\widehat{E}_t\Omega_{t+1}^O$ is the biased expectation of future discount factors. Assuming the difference between the two expectations can be given as

$$\widehat{E}_t\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y} = (1+Bias_t)E_t\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}$$
$$\widehat{E}_t\Omega_{t+1}^O = (1+Bias_t)E_t\Omega_{t+1}^O$$

and the $Bias_t$ follows an exogenous process. Assuming zero initial bias initially the rational and bounded rational model has the same initial steady-state. The previous aging scenario was extended with 0.25% permanent shifts of the bias (see figure 4). It means that the households have optimistic expectation than rational households.

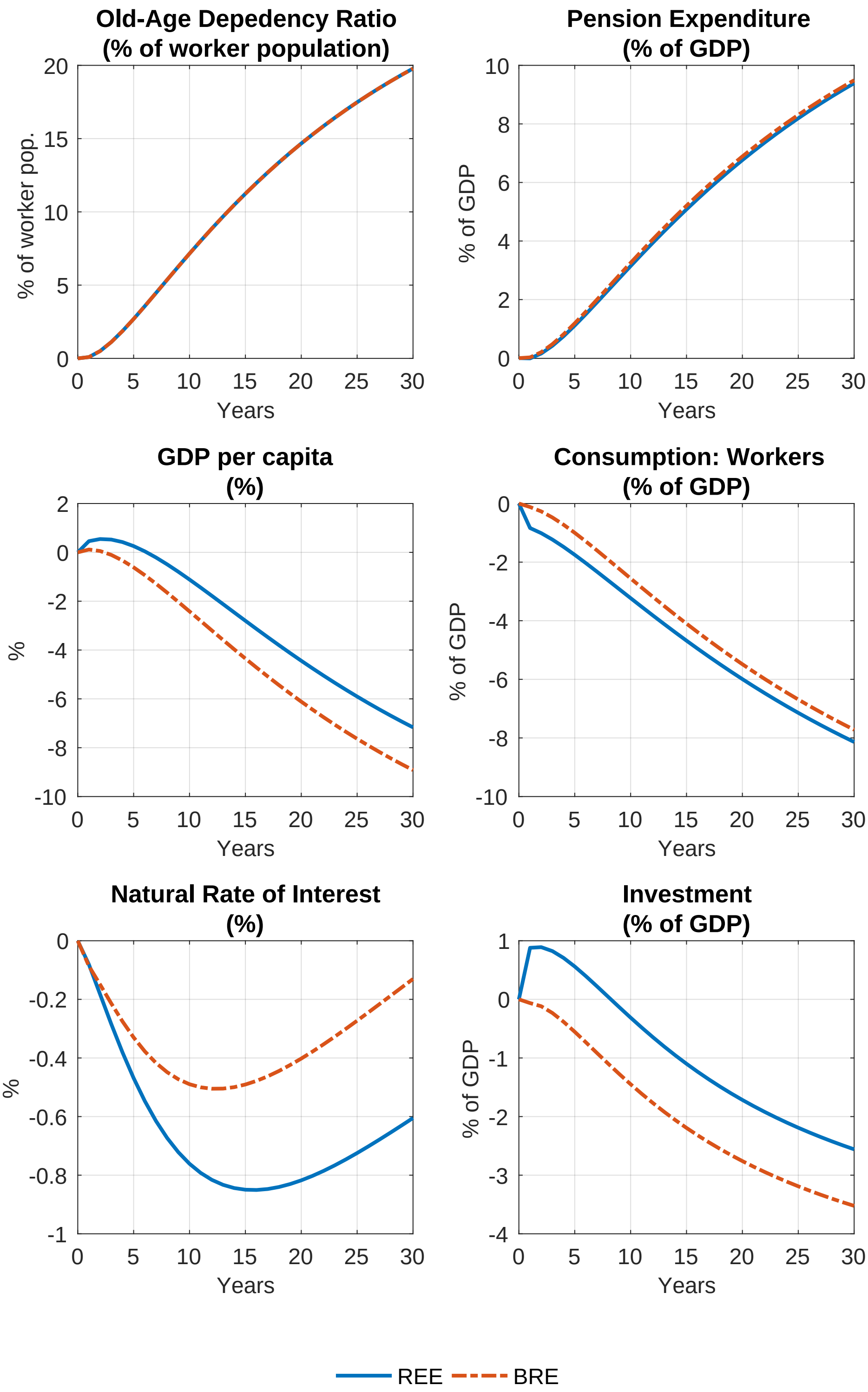

Figure 4: Population Aging and transitional dynamics in bounded rational expectation equilibrium

Compared to the rational equilibrium (REE) the workers are not willing to give up as much consumption in bounded rational equilibrium (BRE), moreover, they take credits to compensate for increasing taxes. The increasing demand on the bonds market exerts additional positive effect on real interest rate, and it does not decrease as much as in rational equilibrium. The interest differentials explain the drop in the investment, and the lower capital level contributes to lower GDP per capita.

A permanent bias could generate significant differences in the new steady-state equilibrium after the population aging (see figure 5). The two models have the same initial equilibrium point (solid lines), but the bias shifts the dynamic IS-curve, because the workers' expectation is more optimistic than their rational self (see red dotted line). The demand curve for young bond is also shifted, however this distortion does not change significantly the position of the curve.

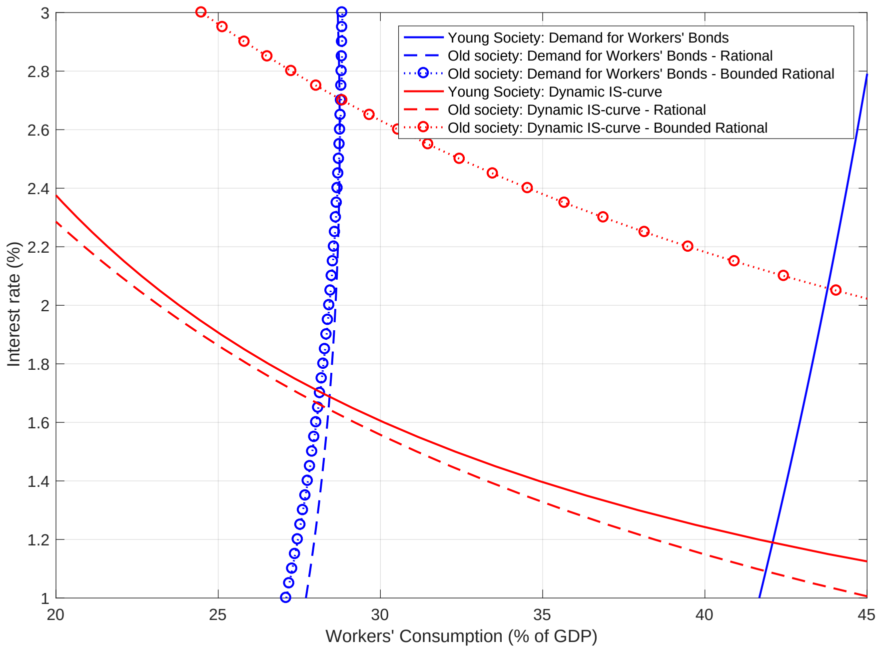

Figure 5: Initial and new steady-state equilibrium: bounded rational expectation equilibrium versus rational expectation equilibrium

In the following subsection I give more formal description for the bounded rationality, and instead of ad-hoc assumption of the distorted expectation channel I describe the level-k thinking as a special case of bounded rationality.

4.1 Bounded rationality and level-k thinking

The level-k thinking has a growing literature in nowadays macroeconomic literature. The concept of level-k thinking is used from Fair and Taylor (1983) and Evans and Ramey (1992; 1995; 1998), these papers compute the rational expectation equilibrium from iterative steps, and as one choses different length for the iteration different equilibrium outcomes can be calculated. Fahri-Werning (2017) and Gabaix (2018) implemented level-k thinking into dynamic stochastic general equilibrium models, however both paper concentrate on only the short run fluctuations and the effect of monetary policy. The main advantages of level-k thinking is the tractability and compatibility with the standard DSGE-models, because it only changes the the expectation operator, and assumes the agents follow the same behavioral equations.

Assuming x as a forward-looking variable and y as another contemporaneous variable we can write any forward looking difference equation in the following way

$$x_t^k = \alpha x_{t+1}^{e,k-1} + \beta f(y_t)$$

where $f(\cdot)$ is a well-defined function of y_t fundamental variables and $|\alpha| < 1$. For the case of $k = 1$ we assume:

$$x_t^1 = \alpha x_{t+1}^{e,0} + \beta f(y_t)$$

where $x_{t+1}^{e,0}$ equal with the initial steady-state value for all t. For higher k we can write the following equations:

$$\begin{aligned} x_t^2 &= \alpha x_{t+1}^{e,1} + \beta f(y_t) \\ x_t^3 &= \alpha x_{t+1}^{e,2} + \beta f(y_t) \\ &\vdots \\ x_t^k &= \alpha x_{t+1}^{e,k-1} + \beta f(y_t) \end{aligned}$$

If one substitutes out the expectation terms, the x_t^k can be expressed as the sum of next period $f(y)$-s and initial value:

$$x_t^k = \alpha^k x_{t+1}^{e,0} + \beta \sum_{n=0}^{k-1} \alpha^n f(y_{t+n}^e)$$

Based on this the equation above, the rational expectation equilibrium (REE) can be interpreted as the special case of bounded rationality. According to the rational expectation assumption, the agents take into account all available information, it formally means $\lim_{k\to\infty} \alpha^k = 0$. x_t^∞ is independent from its initial steady-state value, and it equals with the x_t from rational expectation equilibrium. The size of α and k are crucial to see how the initial conditions affect the dynamics of x. In the next subsection I show the modified equations of the OLG-model, based on the formula above I calculate the level-k consistent expectations and compare the dynamic and steady-state properties of the demographic aging shock.

4.2 Gertler-type OLG with level-k thinking

The main contribution of this paper is that I combine the Gertler-type OLG model with level-k thinking and describe the population aging in non-rational economies. Previously I have shown that the model can be simplified into two main blocks of equations. The first block, the dynamic IS-curve from worker cohort with level-k thinking, can be written as:

$$\frac{\tilde{C}_t^{Y,k}}{MPC_t^{Y,k}} = \frac{\tilde{C}_t^{Y,k}}{\sigma} + \tilde{B}_t^{Y,k}\left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}}\right) + \frac{\omega_t^Y}{1+r_t}\nu\tilde{Y}_t\Omega_{t+1}^{O,e,k} + \frac{1-\omega_t^Y}{1+r_t}\frac{1+s_{t+1}}{1+s_t}\frac{\tilde{C}_{t+1}^{Y,e,k-1}}{MPC_{t+1}^{Y,e,k-1}}$$

where the additional variables are

$$\begin{aligned}
\frac{1}{MPC_t^{Y,k}} &= \frac{1}{\sigma} + (1+r_t)^{\frac{1}{\gamma}-1}\left[(1-\omega_t^Y)\Lambda_t^Y\frac{1}{MPC_{t+1}^{Y,e,k-1}} + \omega_t^Y\Lambda_t^{YO}\frac{1}{MPC_{t+1}^{O,e,k-1}}\right] \\
\frac{1}{MPC_t^{O,k}} &= 1 + (1-\omega_t^O)(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\frac{1}{MPC_{t+1}^{O,e,k-1}} \\
\Omega_t^{O,k} &= 1 + \frac{1-\omega_t^O}{1+r_t}\Omega_{t+1}^{O,e,k-1}
\end{aligned}$$

The level-k thinking changes the role of expectation, and as we have seen before in the case of bounded rationality the expectation operator is biased toward to the initial point. In the appendix I show, that the initial steady-state of the bounded and full rational equilibrium are the same. The bounded rationality distorts the new steady-state equilibrium of the aging society. In appendix I also show the analytical form of the steady-state and compare the two equilibriums.

Since the retiree's discount factor also contains expectation terms, then the retired cohort consumption can be distorted and the demands for the worker bond differs from the rational equilibrium level:

$$\tilde{B}_t^{Y,k} \;=\; \tilde{Debt}_t - (1 - MPC_t^{O,k}\Omega_t^{O,k})\tilde{TR}_t - \frac{(1+r_{t-1})}{1+g_t^N}(1 - MPC_t^{O,k})\left[\tilde{Debt}_{t-1} - (1-\omega_{t-1}^{Y})\tilde{B}_{t-1}^{Y,k}\right]$$

Rest of the equations and variables are the same as in the previous model with rational expectation. The real interest rate is the function of these two equations above, that determines also the expected marginal product of capital in the no-arbitrage condition. Because the level-k consistent real interest rate anchores a level-k consistent expectation of marginal product of capital, the capital accumulation is also consistent with level-k expectation.

4.3 Aging shock and level-k thinking

Compared to rational expectation equilibrium (REE), the bounded rationality (BRE with different level of k) changes the workers' savings attitude (see figure 6). Under rational expectation the workers from the first moment of the aging shock take into account that on the medium-term as the aging will become more advanced, the government will automatically increase the lump-sum taxes to sustain the pension system. Hence, the workers immediately adjust their permanent income, decide to save more and decrease their consumption. In the case of bounded rationality these patterns are different, the young households with lower k are biased toward the initial steady-state consumption, and beyond k period they expect the economy reverts back to the initial state of the economy. Therefore, they are not likely to be saving, as the society start aging because they do not think that the aging permanently changes the long-run position of the economy, moreover to offset the increasing taxes the workers start cumulating domestic credit to smooth out their consumption. Contrary to the conventional prediction of OLG models with rational expectations the increasing credit position exerts a positive pressure on real interest rate. That becomes a strong incentives for the retired households to save even more and finance the government.

The increasing interest rate redistributes the wealth among generations. Initially, the retired households are willing to consume less than in rational expectation equilibrium. On the medium term as the aging become more prevailed they own relatively more savings, and due to the wealth effect the old-age consumption can exceed the rational expectation equilibrium.

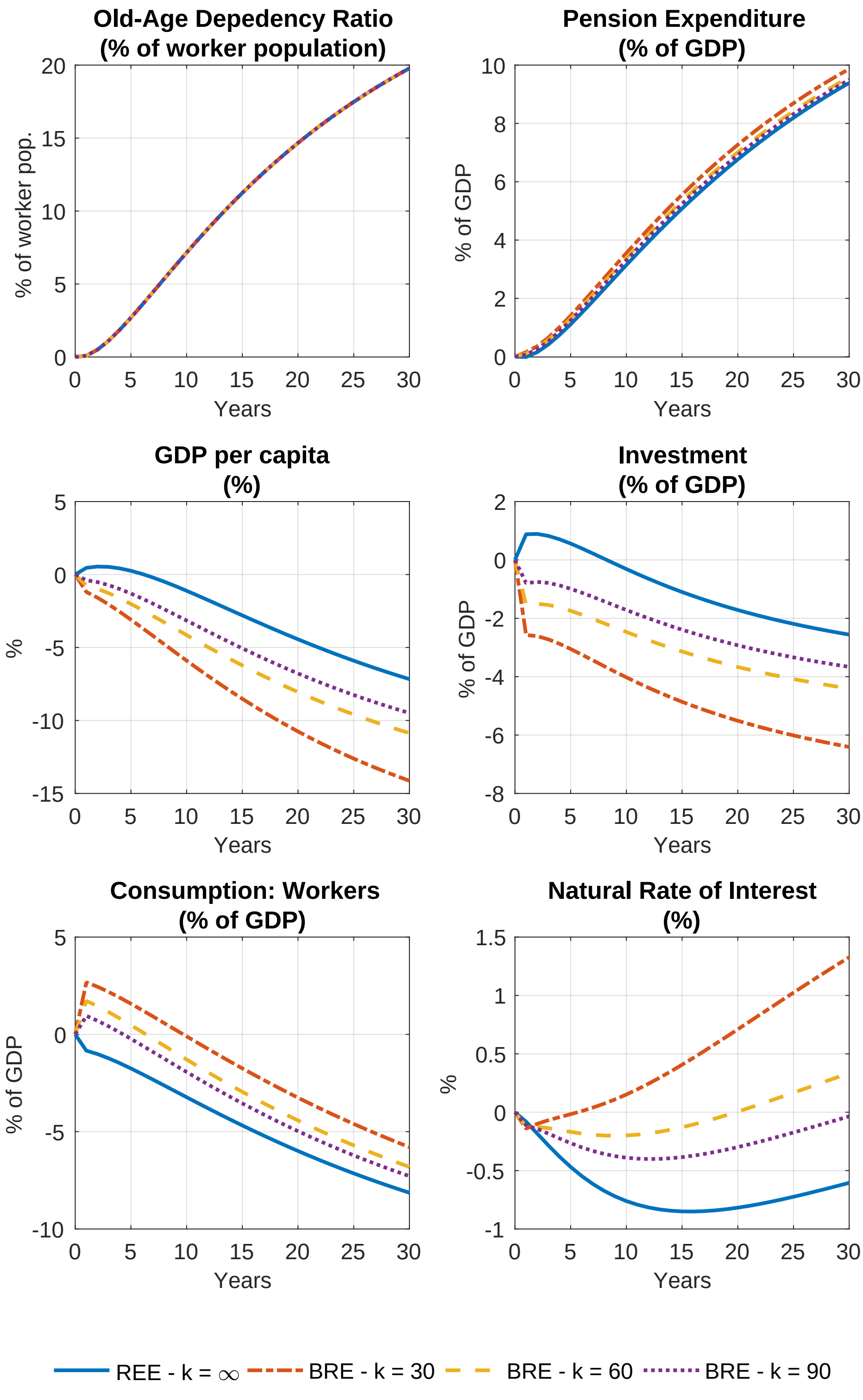

Figure 6: Population Aging and transitional dynamics: Bounded rationality (BRE) versus full rationality (REE)

The bounded rationality generates significant differences in the new steady-state equilibrium after the population aging (see figure 7). The two model has the same initial equilibrium point (solid lines), but the level-k thinking (with finite k) shifts and changes the slope of the dynamic IS-curve, because the workers' expectation is distorted toward the initial steady-state equilibrium point (see red dotted line). The demand curve for young bond is also shifted, however the bounded rationality does not changes significantly the slope of the curve. The reason for the almost unchanged demand curve is the shorter life-time horizon of the retired households. Although the demand curve takes into account the old households' consumption function and the government budget constraint, but the bounded rationality only affects via the expectation of the old-age discount factor that does not change significantly by relatively large ω^O.

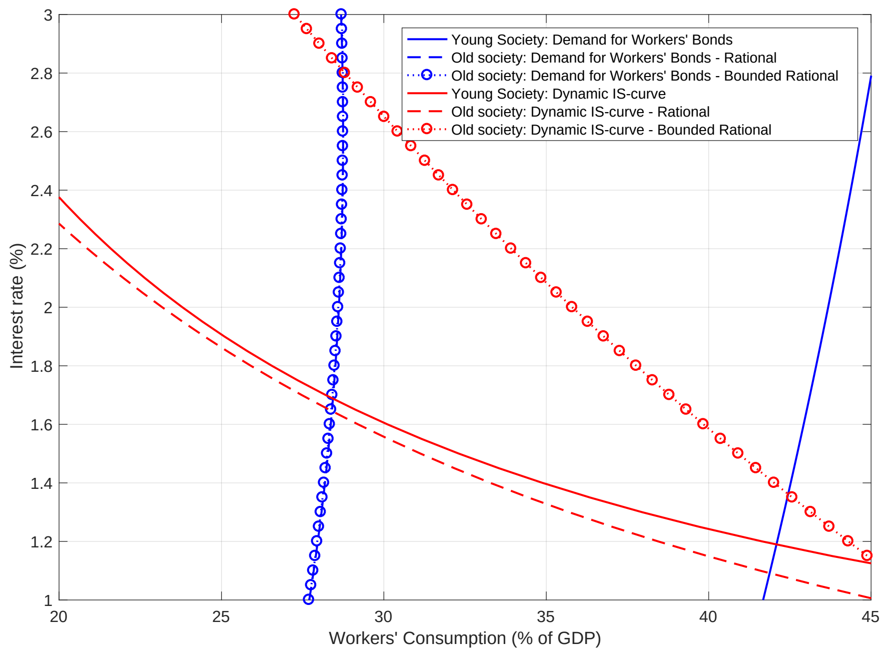

Figure 7: Initial and new steady-state equilibrium: bounded rational expectation equilibrium (k=60) versus rational expectation equilibrium

The bounded rationality sheds light on a possible identification problem of the secular stagnation. While the standard OLG-models have similar prediction about the decreasing natural rate of interest at the time of population aging, under bounded rationality the strength and sign of this relationship is the function of the k. Moreover, the negative relationship is hold only for those country where the k is large that means the agents have a long enough planning horizon, otherwise the aging could generate different dynamics and steady-state in non-rational economies. In the next part I adjust the previous estimation, show some examples how to control on the rational behavior, and check whether the model prediction is consistent with the observed data.

5 Controlling on rational expectations

The bounded rationality models show that in aging societies the natural rate of interest is not necessarily decreasing. In those countries where the agents have biased expectation or concentrate on only the short period of their expected life-time, the interest rate could increase during population aging. In this part I show which proxy I used to control on the countries being 'rational' or 'non-rational', and we can also check, whether in 'rational' societies, like in standard OLG-models, the secular stagnation hypothesis is robust and valid assumption.

The essential question is how to observe or separate the countries being rational or non-rational. I have found two alternative proxies for the selection. The first proxy is the financial literacy indicator 2014 S&P FinLit Survey (Klapper et al, 2014). This survey measured the participants' understanding in risk diversification, inflation, interest rate and compound interest rate calculations, it was sponsored by the S&P and involved all countries in world. The second proxy is the time preference index from the Global Preference Survey (Falk et al, 2018), that is a global representative dataset from 76 countries around the world. The time preference was measured from the combination of a quantitative and qualitative surveys, a series of binary choices and self-assessment about their willingness to wait for the immediate or delayed financial reward. In my perspective, it is probable that the average planning horizon is longer in those countries, where the agents are financially literate or more patient about the future economic pay-offs.

Before the panel estimation, I test whether the negative relationship between old-age dependency ratio and interest rate is stronger if the economy has higher financial literacy index or larger time preference. For this exercise I calculate simple correlations between the country level interest rate and old-age dependency ratio, and create cross-plot figure between the estimated correlation and financial literacy or time preference indicator.

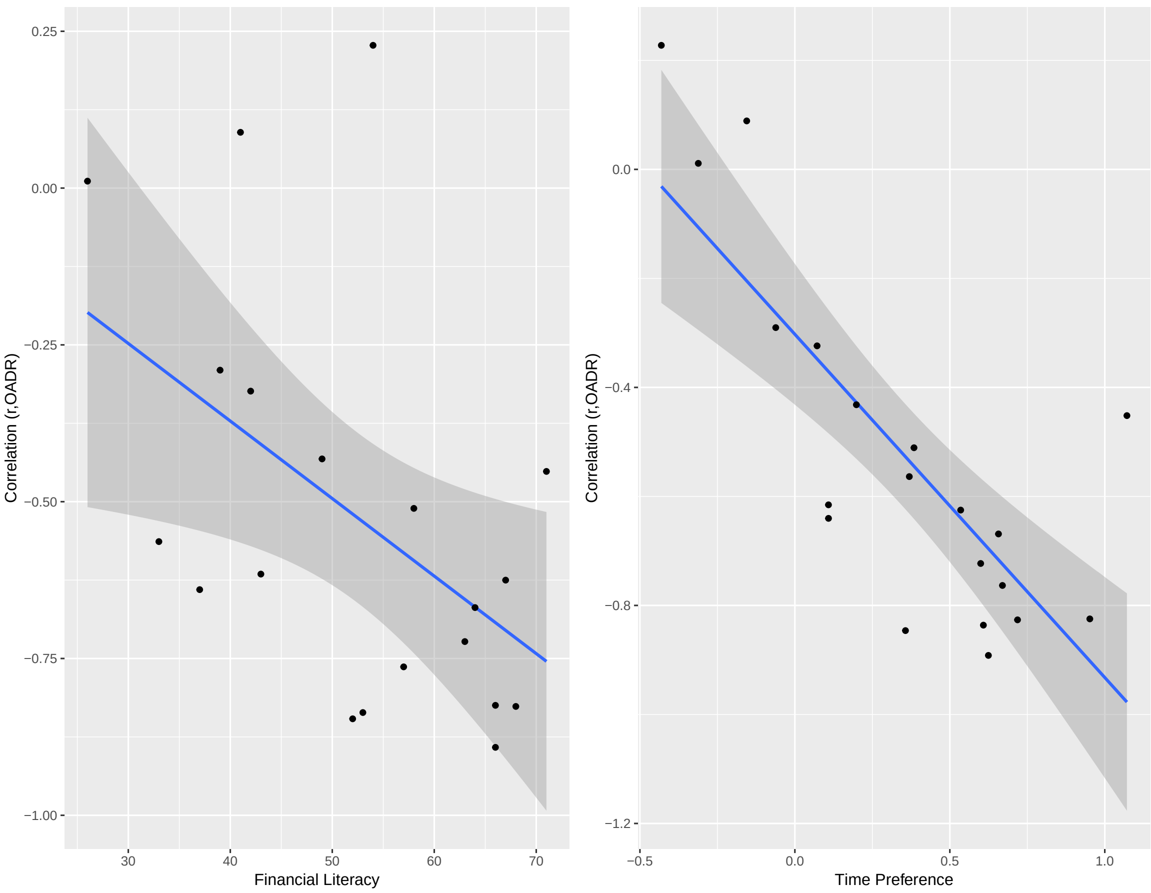

Figure 8: Country-level coefficients with financial literacy or time preference

Based on the figure 8, there is negative relationship among the indicators and the estimated correlation. The co-movement is stronger if I use time preference on x-axis, the correlation between the time preference and estimated correlations is -0.79, while in the other case it is -0.50. These findings support the idea that in those countries, where the households have better understanding of the nature of the financial markets or have longer planning horizon, they might have more savings, which would put negative pressure on domestic long-term real interest rate. In addition, these co-movements also suggests, that these proxies could be a good control for the selection as well.

In the final step, using the same dataset of OECD countries between 1992 and 2016, I estimate the following unbalanced panel with interaction term:

$$r_{it} = \rho \cdot r_{i,t-1} + \alpha_i + \delta_t + (\beta + \xi \cdot D_i) \cdot OADR_{it} + u_{it}$$

where $D_i = 1$ if the selected index for country i is larger than the average, $D_i = 0$ for the rest of the countries.

The results for the estimations are reported in table 2. FE (4) and FE (5) show the results of the first estimation. FE (7) and FE (8) are the static and dynamic panel estimation of the model with financial literacy indicator, FE (9) and FE (10) are the static and dynamic panel estimation of the model with time preference. As before I also estimated robust standard errors and reported them in the parentheses.

	Dependent variable:					
	r_{it}					
	FE (4)	FE (5)	FE (7)	FE (8)	FE (9)	FE (10)
r_{it-1}		0.49		0.48		0.48
		(0.09)***		(0.09)***		(0.09)***
$OADR_{it}$	−0.06	−0.03	0.02	−0.01	0.03	0
	(0.09)	(0.05)	(0.11)	(0.06)	(0.11)	(0.06)
$D_i(\text{FinLit}) \cdot OADR_{it}$			−0.24	−0.08		
			(0.13)*	(0.04)*		
$D_i(\text{GPS}) \cdot OADR_{it}$					−0.29	−0.12
					(0.12)**	(0.04)***
Country FE	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	No	Yes	Yes	Yes
Observations	483	482	483	482	483	482
R^2	0.39	0.56	0.41	0.56	0.41	0.56
Note:						*p<0.1; **p<0.05; ***p<0.01 Robust standard errors in parentheses

Table 2: Estimation with financial literacy and time preference

The interaction terms have a significant negative relationship despite of two-ways estimation. These results suggest that one percentage point permanent increase in old-age dependency ratio ceteris paribus generates 15-29 basis point decrease in the long-term real interest rate, if the country received higher than average score for financial literacy index or households have higher than average time preferences. For those countries where the financial literacy indicator or the time preference is below the average, none of the estimation results in significant relationship. These findings are also consistent with the the prediction of the model with bounded rationality, namely that the negative relationship is not necessarily holds for all countries, only for those where the agents' expectation is consistent with the rational expectations.

6 Summary and conclusion

This paper revises the secular stagnation hypothesis through the lens of bounded rationality with level-k thinking. To the best of my knowledge, this is the first Gertler-type

OLG-model with bounded rationality, that examines the effect of population aging in non-rational economies. It can be shown if the agents' expectations are far from the rational expectations, the demographic changes will not happen with decreasing real interest rate, because in non-rational cases the young households do not save enough to prepare themselves for the longer life-time horizon. This theoretical contribution also gives more insight for the identification strategy, and explains why the panel estimations of relationship between demography and long-term interest rate are not robust for two-ways estimations. In the rest of the paper I added an interaction terms to the panel estimation, that differentiates the countries from being rational or non-rational. The rationality was proxied by the S&P Financial Literacy survey and in another specification by the time preference from the Global Preference Survey. Both of the adjusted estimation have shown that the secular stagnation hypothesis is hold for those countries where the agents' behavior is consistent with the rational expectations that is in-line with the message of the OLG-model with bounded rationality.

References

1. Aksoy, Yunus, Henrique S. Basso, Ron P. Smith and Tobias Grasl (2019): Demographic Structure and Macroeconomic Trends, American Economic Journal: Macroeconomics, vol. 11, no. 1, January 2019, (pp. 193-222)
2. Arellano, Manuel, and Stephen Bond (1991): Some Tests of Specification for Panel Data: Monte Carlo Evidence and an Application to Employment Equations. The Review of Economic Studies, vol. 58, no. 2, 1991, pp. 277?297. JSTOR, JSTOR, www.jstor.org/stable/2297968.
3. Arslanalp, Serkan, Jaewoo Lee and Umang Rawat (2018): Demographics and Interest Rates in Asia, IMF Working Paper, Working Paper No. 18/172
4. Auerbach, Alan J., and Lawrence J. Kotlikoff (1987): Dynamic fiscal policy, Cambridge University Press.
5. Baksa, Daniel, Carolin Nerlich and Zsuzsa Munkácsi (2019): How costly are the pension reform reversals? ECB Working Paper, fothcoming
6. Baksa, Daniel, and Zsuzsa Munkácsi (2016): A Detailed Description of OGRE, the OLG Model. Bank of Lithuania, Working Paper Series, No. 31/2016.
7. Baksa, Daniel, and Zsuzsa Munkácsi (2019): More gray, more volatile? Aging and (optimal) monetary policy. Bank of Lithuania and IMF Working Paper, forthcoming
8. Bielecki, Marcin, Michal Brzoza-Brzezina and Marcin Kolasa (2018): Demographics, monetary policy, and the zero lower bound, NBP Working Papers 284, Narodowy Bank Polski, Economic Research Department.
9. Blanchard, Oliver. J. (1985): Debt, deficits, and finite horizons, Journal of Political Economy, 93, 223-247.
10. Branch, William, George W. Evans and Bruce McGough (2013): Finite Horizon Learning, Chapter 8 in Macroeconomics at the Service of Public Policy, edited by T.J. Sargent and J. Vilmunen, Oxford University Press, 2013.
11. Carvalho, Carlos, and Andrea Ferrero (2014): What Explains Japan's Persistent Deflation?, Manuscript.
12. Carvalho, Carlos, Andrea Ferrero, and Fernanda Nechio (2016): Demographics and Real Interest Rates: Inspecting the Mechanism, Federal Reserve Bank of San Francisco Working Paper 2016-05.
13. Conlisk, John (1996): Why Bounded Rationality?, Journal of Economic Literature, Vol. 34, No. 2. (Jun., 1996), pp. 669-700.

14. Croissant, Yves and Giovanni Millo (2008): Panel Data Econometrics in R: The plm Package, Journal of Statistical Software, July 2008, Volume 27, Issue 2.

15. Eggertsson, Gauti B., Neil R. Mehrotra, and Jacob A. Robbins (2017): A Model of Secular Stagnation: Theory and Quantitative Evaluation, NBER Working Paper 23093, January 2017.

16. Evans, George W. and Gary Ramey (1992): Expectation Calculation and Macroeconomic Dynamics, American Economic Review, 1992, 82, 207?224.

17. Evans, George W. and Gary Ramey (1995): Expectation Calculation, Hyperinflation, and Currency Collapse, in By H. Dixon and N. Rankin, eds., The New Macroeconomics: Imperfect Markets and Policy Effectiveness, Cambridge University Press

18. Evans, George W., Seppo Honkapohja and Kaushik Mitra (2017): Expectations, Stagnation and Fiscal Policy CEPR, Discussion Paper Series, DP11428

19. Evans, George W. and Gary Ramey (1998): Calculation, Adaptation, and Rational Expectations, Marcoeconomic Dynamics, 1998, 2, 156-182.

20. Evans, George W., Seppo Honkapohja and Kaushik Mitra (2010): Does Ricardian Equivalence Hold When Expectations are not Rational?, Bank of Finland Research Discussion Paper No. 13/2010

21. Falk, A., Becker, A., Dohmen, T., Enke, B., Huffman, D., & Sunde, U. (2018). Global evidence on economic preferences. Quarterly Journal of Economics, 133 (4), 1645-1692.

22. Fair, Ray and John Taylor (1983): Solutions and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models, Econometrica, 1983, 51, 1169-1189.

23. Favero, Carlo A., and Vincenzo Galasso (2015): Demographics and the Secular Stagnation Hypothesis in Europe, CEPR Discussion Paper Series No. 10887, October 2015.

24. Farhi, Emmanuel and Iván Werning (2017): Monetary Policy, Bounded Rationality, and Incomplete Markets, September 2017, manuscript

25. Ferrero, Guiseppe, Marco Gross, and Stefano Neri (2017): On Secular Stagnation and Low Interest Rates: Demography Matters, ECB Working Paper No 2088, July 2017.

26. Gabaix, Xavier (2018): A Behavioral New Keynesian Model, Harvard working paper, Harvard 2017

27. Gabaix, Xavier (2017): Behavioral Macroeconomics Via Sparse Dynamic Programming, Manuscript, March, 2017

28. Gagnon, Etienne, Benjamin K. Johannsen, and David Lopez-Salido (2016): Understanding the New Normal: The Role of Demographics, Finance and Economics Discussion Series 2016-080. Washington: Board of Governors of the Federal Reserve System

29. Gertler, Mark (1999): Government Debt and Social Security in a Life-Cycle Economy, Carnegie-Rochester Conference Series on Public Policy Vol. 50(1): 61-110.

30. Jones, Callum (2018): Aging, Secular Stagnation and the Business Cycle, IMF Working Paper, WP/18/67

31. Kara, Engin, and Leopold von Thadden (2010): Interest Rate Effects of Demographic Changes in a New-Keynesian Life-Cycle Framework, ECB Working Paper Series, No 1273/December 2010.

32. Klapper, Leora, Annamaria Lusardi and Peter van Oudheusden (2014): Financial Literacy Around the World: Insights from the Standard & Poors Ratings Services, Global Financial Literacy Survey, The George Washington University, School of Business, Global Financial Literacy Excellence Center

33. Kumhof, Michael, Douglas Laxton, Dirk Muir and Susanna Mursula (2010): The Global Integrated Monetary and Fiscal Model (GIMF) - Theoretical Structure, IMF Working Paper, WP/10/34

34. Lovo, Stefano (2000): Infinitely lived representative agent exchange economy with myopia, Manuscript, February 2010

35. Park, Hyeon and James Feigenbaumz (2018): Bounded Rationality, Lifecycle Consumption, and Social Security, Journal of Economic Behavior & Organization, Elsevier, vol. 146(C), pages 65-105.

36. Sims, Christopher A. (2003): Implications of Rational Inattention. Journal of Monetary Economics, 50(3), 665-690.

37. Summers, Lawrence (2014): U.S. Economic Prospects: Secular Stagnation, Hysteresis, and the Zero Lower Bound, Business Economics Vol. 49, No. 2, Keynote Address at the NABE Policy Conference, February 2014.

38. Yaari, M. (1965): Uncertain lifetime, life insurance, and the theory of the consumer, Review of Economic Studies, 32, 137-150.

Appendix

The appendix contains the summary of the model equations, the derivations of the simplified model, the parameters and the initial steady-state ratios, the steady-state calculations of bounded rationality equilibrium.

Appendix 1: List of the model equations

Demography:

$$
\begin{aligned}
s_t &= \frac{(1-\omega^O_{t-1})}{(1-\omega^Y_{t-1}+n_t)}s_{t-1} + \frac{\omega^Y_{t-1}}{(1-\omega^Y_{t-1}+n_t)} \\
s^Y_t &= \frac{1}{1+s_t} \\
1+g^{N,Y}_t &= 1-\omega^Y_{t-1}+n_t \\
1+g^{N,O}_t &= (1-\omega^O_{t-1}) + \frac{\omega^Y_{t-1}}{s_{t-1}} \\
1+g^N_t &= (1+g^{N,Y}_t)\frac{1+s_t}{1+s_{t-1}}
\end{aligned}
$$

Retired households:

$$
\begin{aligned}
\tilde{C^O_t} &= MPC^O_t \tilde{TR}_t \Omega^O_t + MPC^O_t \frac{(1+r_{t-1})}{1+g^N_t}\left(\omega^Y_{t-1}\tilde{B^Y_{t-1}} + \tilde{B^O_{t-1}}\right) \\
\Omega^O_t &= 1 + E_t \frac{1-\omega^O_t}{1+r_t}\Omega^O_{t+1} \\
\frac{1}{MPC^O_t} &= 1 + E_t(1-\omega^O_t)(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\frac{1}{MPC^O_{t+1}}
\end{aligned}
$$

Worker households:

$$
\begin{aligned}
\tilde{C}_t^Y &= MPC_t^Y \tilde{Inc}_t + MPC_t^Y \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t^N} \tilde{B}_{t-1}^Y \\
\tilde{Inc}_t &= w_t s_t^Y + \tilde{Profit}_t - \tilde{Tax}_t + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc}_{t+1} \\
\frac{\tilde{C}_t^Y}{s_t^Y - \tilde{L}_t} &= \frac{\sigma}{1-\sigma} w_t \\
\frac{1}{MPC_t^Y} &= \frac{1}{\sigma} + E_t (1+r_t)^{\frac{1}{\gamma}-1} \left[(1-\omega_t^Y) \Lambda_t^Y \frac{1}{MPC_{t+1}^Y} + \omega_t^Y \Lambda_t^{YO} \frac{1}{MPC_{t+1}^O} \right] \\
\Lambda_t^Y &= \beta^{\frac{1}{\gamma}} \left(E_t \frac{w_{t+1}}{w_t} \right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \\
\Lambda_t^{YO} &= \left\{ \frac{\beta}{\sigma} \right\}^{\frac{1}{\gamma}} \left(\frac{1}{\frac{\sigma}{1-\sigma} w_t} \right)^{(1-\sigma)\left(1-\frac{1}{\gamma}\right)} \\
\tilde{B}_t^Y &= w_t \tilde{L}_t + \tilde{Profit}_t - \tilde{Tax}_t - \tilde{C}_t^Y + \frac{(1+r_{t-1})}{1+g_t^N} (1-\omega_{t-1}^Y) \tilde{B}_{t-1}^Y
\end{aligned}
$$

Firms:

$$
\begin{aligned}
\tilde{Y}_t &= A_t \left(\frac{\tilde{K}_{t-1}}{1+g_t^N} \right)^{\alpha} \tilde{L}_t^{1-\alpha} \\
\tilde{K}_t &= \tilde{Inv}_t + (1-\delta) \frac{\tilde{K}_{t-1}}{1+g_t^N} \\
w_t &= (1-\alpha) \frac{\tilde{Y}_t}{\tilde{L}_t} \\
1+r_t &= E_t \alpha (1+g_{t+1}^N) \frac{\tilde{Y}_{t+1}}{\tilde{K}_t} + (1-\delta) \\
\tilde{Profit}_t &= \tilde{Y}_t - w_t \tilde{L}_t - \tilde{Inv}_t
\end{aligned}
$$

Fiscal policy:

$$
\begin{aligned}
\tilde{Debt}_t &= \tilde{TR}_t + \tilde{Gov}_t - \tilde{Tax}_t + \frac{(1+r_{t-1})}{1+g_t} \tilde{Debt}_{t-1} \\
\tilde{TR}_t &= \nu \frac{\omega_{t-1}^Y}{1+g_t^N} w_{t-1} \tilde{L}_{t-1} + \frac{(1-\omega_{t-1}^O)}{1+g_t^N} \tilde{TR}_{t-1}
\end{aligned}
$$

Equilibrium conditions:

$$
\begin{aligned}
\tilde{Debt}_t &= \tilde{B}_t^Y + \tilde{B}_t^O \\
\tilde{Y}_t &= \tilde{C}_t^Y + \tilde{C}_t^O + \tilde{Inv}_t + \tilde{Gov}_t
\end{aligned}
$$

Appendix 2: Simplified model

The life-time income can be expressed as:

$$\tilde{Inc}_t = w_t s_t^Y + \tilde{Profit}_t - \tilde{Tax}_t + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc}_{t+1}$$

Plug in $\tilde{Profit}_t$:

$$\tilde{Inc}_t = w_t s_t^Y + \tilde{Y}_t - \tilde{Inv}_t - w_t \tilde{L}_t - \tilde{Tax}_t + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc}_{t+1}$$

Based on labor supply curve we can substitute out $w_t s_t^Y - w_t \tilde{L}_t$ with $\frac{1-\sigma}{\sigma} \tilde{C}_t^Y$, and in the next step we can also substitute out $\tilde{Y}_t$ with demand components in the goods market equilibrium:

$$\tilde{Inc}_t = \frac{1}{\sigma} \tilde{C}_t^Y + \tilde{C}_t^O + \tilde{Gov}_t - \tilde{Tax}_t + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \tilde{Inc}_{t+1}$$

From the workers' consumption function we can express $\tilde{Inc}_t$:

$$\tilde{Inc}_t = \frac{\tilde{C}_t^Y}{MPC_t^Y} - \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t^N} \tilde{B}_{t-1}^Y$$

Plugging it back into the previous equation, and substituting out $\tilde{Gov}_t - \tilde{Tax}_t$ from the government budget constraint we get:

$$\frac{\tilde{C}_t^Y}{MPC_t^Y} - \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t^N} \tilde{B}_{t-1}^Y = \frac{1}{\sigma} \tilde{C}_t^Y + \tilde{C}_t^O + \tilde{Debt}_t - \tilde{TR}_t - \frac{(1+r_{t-1})}{1+g_t} \tilde{Debt}_{t-1} + \\ + E_t \frac{\omega_t^Y \nu w_t \tilde{L}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \left(\frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y} - \frac{(1+r_t)(1-\omega_t^Y)}{1+g_{t+1}^N} \tilde{B}_t^Y \right)$$

For the next steps we need to use the bonds market equilibrium and the retireds' budget constraint:

$$\begin{aligned} \tilde{Debt}_t &= \tilde{B}_t^Y + \tilde{B}_t^O \\ \tilde{C}_t^O + \tilde{B}_t^O &= \tilde{TR}_t + \frac{1+r_{t-1}}{1+g_t^N} \left(\omega_{t-1}^Y \tilde{B}_{t-1}^Y + \tilde{B}_{t-1}^O \right) \end{aligned}$$

Substituting out $\tilde{Debt}_t$ from the bonds market equilibrium and $\tilde{C}_t^O - \tilde{TR}_t$ from the retired budget constraint, we can get the "dynamic IS-curve":

$$\frac{\tilde{C}_t^Y}{MPC_t^Y} = \frac{\tilde{C}_t^Y}{\sigma} + \tilde{B}_t^Y \left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}} \right) + E_t \frac{\omega_t^Y}{1+r_t} \nu \alpha \tilde{Y}_t \Omega_{t+1}^O + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}$$

To determine the demand function for the workers' bond, we need to rearrange the workers' budget constraint, substitute out the $\tilde{Profit}_t$ and $\tilde{Y}_t$ from the goods market equilibrium

$$\begin{aligned}
\tilde{C_t^Y} + \tilde{B_t^Y} + \tilde{Tax_t} &= w_t\tilde{L}_t + \tilde{Profit_t} + \frac{(1+r_{t-1})}{1+g_t^N}(1-\omega_{t-1}^Y)\tilde{B_{t-1}^Y} \\
\tilde{B}_t^Y &= \tilde{C}_t^O + \tilde{Gov}_t - \tilde{Tax}_t + \frac{(1+r_{t-1})}{1+g_t^N}(1-\omega_{t-1}^Y)\tilde{B}_{t-1}^Y
\end{aligned}$$

As a next step we can substitute out the retirees' consumption from their consumption function:

$$\begin{aligned}
\tilde{B}_t^Y &= MPC_t^O\tilde{TR}_t\Omega_t^O + MPC_t^O\frac{(1+r_{t-1})}{1+g_t^N}\left(\omega_{t-1}^Y\tilde{B}_{t-1}^Y + \tilde{B}_{t-1}^O\right) + \\
&\quad + \tilde{Gov}_t - \tilde{Tax}_t + \frac{(1+r_{t-1})}{1+g_t^N}(1-\omega_{t-1}^Y)\tilde{B}_{t-1}^Y
\end{aligned}$$

We can substitute out $\tilde{B}_t^O$ from the bonds market equilibrium, and from the government budget constraint we can express $\tilde{Gov}_t - \tilde{Tax}_t$. Rearranging the equation we got the demand function for the workers' bond:

$$\tilde{B}_t^Y = \tilde{Debt}_t - (1 - MPC_t^O\Omega_t^O)\tilde{TR}_t - \frac{(1+r_{t-1})}{1+g_t^N}(1-MPC_t^O)\left[\tilde{Debt}_{t-1} - (1-\omega_{t-1}^Y)\tilde{B}_{t-1}^Y\right]$$

Appendix 3: Parameters and initial steady-state ratios

Name	Notation	Value	Comments
Demography			
Probability of retirement	ω^Y	0.02	Annual frequency
Probability of death	ω^O	0.08	Annual frequency
Fertility rate	n	0.04	Annual frequency
Households			
Discount rate	β	0.99	Annual frequency
Weight of consumption utility	σ	0.6	Only for workers
Inverse of intertemporal elasticity	γ	2	Same for both cohorts
Firms			
Capital share	α	0.3	-
Capital depreciation	δ	0.1	Annual frequency
Fiscal policy			
Replacement rate	ν	0.7463	Calibrated from $\frac{TR}{Y} = 0.1$
Gov. cons. in % of GDP	$\frac{Gov}{Y}$	0.2	-
Debt in % of GDP	$\frac{Debt}{Y}$	0.5	-

Appendix 4: Bounded rationality and steady-state calculations

I assume level-k thinking for the workers, level-l thinking for the retired. Based on the formula we can rewrite the equation of MPC^O:

$$\frac{1}{MPC_t^{O,l}} = \sum_{i=1}^{l} \beta^{\frac{i-1}{\gamma}} \prod_{h=1}^{i-1} (1-\omega_{t+h-1}^O)(1+r_{t+h-1})^{\frac{1}{\gamma}-1} + \beta^{\frac{l}{\gamma}} \prod_{h=1}^{l} (1-\omega_{t+h-1}^O)(1+r_{t+h-1})^{\frac{1}{\gamma}-1} \frac{1}{MPC^{O,*}}$$

where $MPC^{O,*}$ is the initial (original steady-state) value of the MPC^O. And for the Ω^O

$$\Omega_t^{O,l} = \sum_{i=1}^{l} \prod_{h=1}^{i-1} \frac{1-\omega_{t+h-1}^O}{1+r_{t+h-1}} + \prod_{h=1}^{l} \frac{1-\omega_{t+h-1}^O}{1+r_{t+h-1}} \Omega_t^{O,*}$$

where $\Omega^{O,*}$ is the initial (original steady-state) value of the Ω^O.

The MPC^Y can be written as

$$\begin{aligned}\frac{1}{MPC_t^{Y,k}} &= \sum_{i=1}^{k} \prod_{h=1}^{i-1} (1-\omega_{t+h-1}^Y)(1+r_{t+h-1})^{\frac{1}{\gamma}-1} \Lambda_{t+h-1}^Y \left(\frac{1}{\sigma} + \omega_{t+i-1}^Y (1+r_{t+i-1})^{\frac{1}{\gamma}-1} \Lambda_{t+i-1}^{YO} \frac{1}{MPC_{t+i}^{O,k}} \right) \\ &+ \prod_{h=1}^{k} (1-\omega_{t+h-1}^Y)(1+r_{t+h-1})^{\frac{1}{\gamma}-1} \Lambda_{t+h-1}^Y \frac{1}{MPC^{Y,*}}\end{aligned}$$

where $MPC^{Y,*}$ is the initial (steady-state) value of the MPC^Y. The dynamic IS-curve can be given by the following:

$$\left(\frac{1}{MPC_t^Y} - \frac{1}{\sigma}\right) \tilde{C}_t^Y = \tilde{B}_t^Y \left(1 - \frac{(1-\omega_t^Y)^2}{1+g_{t+1}^{N,Y}}\right) + E_t \frac{\omega_t^Y \nu \alpha \tilde{Y}_t \Omega_{t+1}^O}{1+r_t} + E_t \frac{1-\omega_t^Y}{1+r_t} \frac{1+s_{t+1}}{1+s_t} \frac{\tilde{C}_{t+1}^Y}{MPC_{t+1}^Y}$$

If we rearrange and express $\tilde{C}_t^Y$ as the function of forward-looking terms:

$$\begin{aligned}\tilde{C}_t^{Y,k} &= \sum_{i=1}^{k} \left[\prod_{h=1}^{i-1} \frac{\sigma MPC_{t+h-1}^Y}{\sigma - MPC_{t+h-1}^Y} \frac{(1-\omega_{t+h-1}^Y)(1+s_{t+h})}{(1+r_{t+h-1})(1+s_t)} \frac{1}{MPC_{t+h}^Y} \right] \frac{\sigma MPC_{t+i-1}^Y}{\sigma - MPC_{t+i-1}^Y} \left(\tilde{B}_{t+i-1}^Y \left(1 - \frac{(1-\omega_{t+i-1}^Y)^2}{1+g_{t+i}^{N,Y}}\right) + \right. \\ &+ \left. \frac{\omega_{t+i}^Y \nu \alpha \tilde{Y}_{t+i-1} \Omega_{t+i}^O}{1+r_{t+i-1}} \right) + \left[\prod_{h=1}^{k} \frac{\sigma MPC_{t+h-1}^Y}{\sigma - MPC_{t+h-1}^Y} \frac{(1-\omega_{t+h-1}^Y)(1+s_{t+h})}{(1+r_{t+h-1})(1+s_t)} \frac{\tilde{C^Y}^*}{MPC_{t+h}^Y} \right]\end{aligned}$$

By the steady-state calculations the products are simplified into geometric sums, but all steady-state variables that have expectation terms depend on the initial steady-state values also.

The retireds MPC^O in the steady-state can be given as

$$
\begin{aligned}
\frac{1}{MPC^{O,l}} &= \sum_{i=1}^{l}\left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{i-1} + \left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}\frac{1}{MPC^{O,*}} \\
\frac{1}{MPC^{O,l}} &= \frac{1-\left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}}{1-(1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}} + \left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}\frac{1}{MPC^{O,*}} \\
MPC^{O,l} &= \left(\frac{1-\left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}}{1-(1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}} + \left((1-\omega^O)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\right)^{l}\frac{1}{MPC^{O,*}}\right)^{-1}
\end{aligned}
$$

By rational expectation equilibrium the $l \to \infty$ and $k \to \infty$, then MPC^O is independent from $MPC^{O,*}$. Despite of bounded rationality, we got the same results if we assume $MPC^{O,*} = MPC^O$. For the initial steady-state calibration, where $MPC^{O,*} = MPC^O$ condition is satisfied, the rational expectation equilibrium is valid solution of the bounded rational equilibrium. This can be applied for the other forward looking equations also.

The discount factor can be written as

$$
\begin{aligned}
\Omega^{O,l} &= \sum_{i=1}^{l}\left(\frac{1-\omega^O}{1+r}\right)^{i-1} + \left(\frac{1-\omega^O}{1+r}\right)^{l}\Omega^{O,*} \\
\Omega^{O,l} &= \frac{1-\left(\frac{1-\omega^O}{1+r}\right)^{l}}{1-\frac{1-\omega^O}{1+r}} + \left(\frac{1-\omega^O}{1+r}\right)^{l}\Omega^{O,*}
\end{aligned}
$$

The bonds market equilibrium in the steady-state can be expressed as the following:

$$
\tilde{B^Y} = \frac{\tilde{Debt}\left(1-\frac{(1+r)}{1+g}(1-MPC^{O,l})\right) + (MPC^{O,l}\Omega^{O,O,l}(r)-1)\tilde{TR}}{1-\frac{(1+r)}{1+g}(1-MPC^{O,l})(1-\omega^Y)}
$$

By the young households' equation we need to differentiate the marginal propensity to consume and discount factor since the young and old have different level of thinking. The workers' MPC^Y is the following

$$
\begin{aligned}
\frac{1}{MPC^{Y,k}} &= \left(\frac{1}{\sigma}+\frac{\omega^Y(1+r)^{\frac{1}{\gamma}-1}\Lambda^{YO}}{MPC^{O,k}}\right)\sum_{i=1}^{k}\left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{i-1} + \left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}\frac{1}{MPC^{Y,*}} \\
\frac{1}{MPC^{Y,k}} &= \left(\frac{1}{\sigma}+\frac{\omega^Y(1+r)^{\frac{1}{\gamma}-1}\Lambda^{YO}}{MPC^{O,k}}\right)\frac{1-\left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}}{1-\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)} + \left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}\frac{1}{MPC^{Y,*}} \\
\frac{1}{MPC^{Y,k}} &= \left(\left(\frac{1}{\sigma}+\frac{\omega^Y(1+r)^{\frac{1}{\gamma}-1}\Lambda^{YO}}{MPC^{O,k}}\right)\frac{1-\left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}}{1-\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)} + \left(\Lambda^Y(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\right)^{k}\frac{1}{MPC^{Y,*}}\right)^{-1}
\end{aligned}
$$

Based on the workers' consumption function we can express the steady-state version of workers' consumption function:

$$\begin{aligned}\tilde{C}^{Y,k} &= \frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\left(\tilde{B}^{Y}\left(1-\frac{(1-\omega^{Y})^{2}}{1+g^{N,Y}}\right)+\frac{\omega^{Y}\nu\alpha\tilde{Y}\Omega^{O,k}}{1+r}\right)\sum_{i=1}^{k}\left[\frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{1}{MPC^{Y,k}}\right]^{i-1}+\\ &+ \left[\frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{\tilde{C}^{Y,*}}{MPC^{Y,k}}\right]^{k}\end{aligned}$$

After some rearranging:

$$\begin{aligned}\tilde{C}^{Y,k} &= \frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\left(\tilde{B}^{Y}\left(1-\frac{(1-\omega^{Y})^{2}}{1+g^{N,Y}}\right)+\frac{\omega^{Y}\nu\alpha\tilde{Y}\Omega^{O,k}}{1+r}\right)\frac{1-\left[\frac{\sigma MPC^{Y,k}}{\sigma-MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{1}{MPC^{Y,k}}\right]^{k}}{1-\frac{\sigma MPC^{Y,k}}{\sigma-MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{1}{MPC^{Y,k}}}+\\ &+ \left[\frac{\sigma MPC^{Y,k}}{\sigma - MPC^{Y,k}}\frac{1-\omega^{Y}}{1+r}\frac{1}{MPC^{Y,k}}\right]^{k}\tilde{C}^{Y,*}\end{aligned}$$